

ON THE ASCOLI PROPERTY FOR LOCALLY CONVEX SPACES AND TOPOLOGICAL GROUPS

SAAK GABRIYELIAN

ABSTRACT. We characterize Ascoli spaces by showing that a Tychonoff space X is Ascoli iff the canonical map from the free locally convex space $L(X)$ over X into $C_k(C_k(X))$ is an embedding of locally convex spaces. We prove that an uncountable direct sum of non-trivial locally convex spaces is not Ascoli, and a direct countable sum E of metrizable locally convex spaces is Ascoli iff $E = \varphi$. If $X = \varinjlim X_n$ is the inductive limit of a sequence $\{X_n\}_{n \in \omega}$ of metrizable groups such that X_n is closed in X_{n+1} for every $n \in \omega$, then the following assertions are equivalent: (i) there is $m \in \omega$ such that X_n is open in X_{n+1} for every $n \geq m$ or all the X_n are locally compact, (ii) X is sequential; (iii) X is an Ascoli space. Consequently a strict (LF) -space E is Ascoli iff E is a Fréchet space or $E = \varphi$. If a c_0 -barrelled space X is weakly Ascoli, then X is linearly isomorphic to a dense subspace of \mathbb{R}^Γ for some Γ . Consequently a Fréchet space E is weakly Ascoli iff $E = \mathbb{R}^N$ for some $N \leq \omega$. We prove that the weak* dual space of a Banach space E is Ascoli iff E is finite-dimensional.

1. INTRODUCTION.

Being motivated by the classical Ascoli theorem we introduced in [3] Ascoli spaces. A Tychonoff space X is Ascoli if and only if every compact subset of the space $C_k(X)$ of continuous functions $C(X)$ on X endowed with the compact-open topology is equicontinuous. The following diagram gathers some of the most important properties generalizing metrizability:

$$\text{metric} \implies \text{Fréchet-Urysohn} \implies \text{sequential} \implies k\text{-space} \implies k_{\mathbb{R}}\text{-space} \implies \text{Ascoli space},$$

and none of these implications is reversible. The Ascoli property for general topological spaces, function spaces and Banach spaces with the weak topology has been studied recently in [2, 3, 8, 9, 10].

In Section 2 we characterize Ascoli spaces using the notion of free locally convex spaces. The class of free locally convex spaces $L(X)$ over a Tychonoff space X was introduced by Markov [16] and intensively studied over the last half-century, see [13, 22, 27]. If X is a k -space, Flood [5] and Uspenskiĭ [26] proved that the canonical map Δ_X from $L(X)$ to $C_k(C_k(X))$ is an embedding of locally convex spaces (for all relevant definitions see Section 2). Our characterization shows that Δ_X is an embedding (if and) only if X is Ascoli.

Theorem 1.1. *A Tychonoff space X is Ascoli if and only if the canonical map $\Delta_X : L(X) \rightarrow C_k(C_k(X))$ is an embedding of locally convex spaces.*

If $X = D$ is a countable infinite discrete space, then the space $L(D)$ coincides with the direct sum φ of countably many copies of \mathbb{R} endowed with the box topology. It is well-known that φ is a sequential non-Fréchet-Urysohn space, hence φ is Ascoli. However, if D is an *uncountable* discrete space, then the situation changes: $L(D)$ is not an Ascoli space. This result was proved in [2], we give an elementary direct proof of a more general result, see Theorem 3.3 below. For uncountable direct sums of locally convex spaces we obtain the following result proved in Section 3.

2000 *Mathematics Subject Classification.* Primary 46A03; Secondary 54A25, 54D50.

Key words and phrases. the Ascoli property, free locally convex space, direct sum, inductive limit, compactly barrelled space .

Theorem 1.2. *Let $E := \bigoplus_{i \in \kappa} E_i$ be the direct sum of an uncountable family $\{E_i\}_{i \in \kappa}$ of non-trivial locally convex spaces and let τ be either the locally convex direct sum topology \mathcal{T}_{lc} or the box topology \mathcal{T}_b on E . Then (E, τ) is not an Ascoli space and has uncountable tightness.*

Theorem 1.2 motivates the following question: When the locally convex direct sum E of a countable family $\{E_n\}_{n \in \omega}$ of non-trivial Ascoli locally convex spaces is an Ascoli space? It turns out that if all the E_n are metrizable, then E is Ascoli if and only if $E = \varphi$, see Corollary 4.6. This result easily follows from the following general theorem proved in Section 4.

Theorem 1.3. *Let $X = \varinjlim X_n$ be the inductive limit of a sequence $\{X_n\}_{n \in \omega}$ of metrizable groups such that X_n is closed in X_{n+1} for every $n \in \omega$. Then the following assertions are equivalent:*

- (i) *there is $m \in \omega$ such that X_n is open in X_{n+1} for every $n \geq m$ or all X_n are locally compact;*
- (ii) *X is a sequential space;*
- (iii) *X is an Ascoli space.*

For strict (LF) -spaces Theorem 1.3 implies

Corollary 1.4. *Let $E = \varinjlim E_n$ be a strict (LF) -space. Then E is Ascoli if and only if $E = E_m$ for some $m \in \omega$ or $E = \varphi$.*

In [23] Shirai proved that the space $\mathcal{D}(\Omega)$ of test functions over an open subset Ω of \mathbb{R}^n is not sequential. Since $\mathcal{D}(\Omega)$ is a strict (LF) -space and is not metrizable, Corollary 1.4 implies a stronger result:

Corollary 1.5. *$\mathcal{D}(\Omega)$ is not an Ascoli space.*

Let X be a locally convex space and X' be its topological dual space. If X endowed with the weak topology $\sigma(X, X')$ is an Ascoli space we shall say that X is *weakly Ascoli*. Recall that X is called *c_0 -barrelled* if every $\sigma(X', X)$ -null sequence in X' is equicontinuous. Every barrelled space and hence every Fréchet space is c_0 -barrelled.

Theorem 1.6. *If a c_0 -barrelled space X is weakly Ascoli, then X is linearly isomorphic to a dense subspace of \mathbb{R}^Γ , where Γ is a Hamel base of X' .*

Corollary 1.7. *A locally convex space X is a weakly Ascoli complete c_0 -barrelled space if and only if $X = \mathbb{R}^\Gamma$ for some Γ . In particular, a Fréchet space E is weakly Ascoli if and only if $E = \mathbb{R}^N$ for some $N \leq \omega$.*

Note that the last assertion of Corollary 1.7 generalizes Theorem 1.6 of [10] by removing the condition on a Fréchet space E to be a quojection. Our proof of the following result essentially differs from its proof in [2, Theorem 6.1.1], see Remark 5.4.

Corollary 1.8 ([2]). *For a metrizable locally convex space (X, τ) the following assertions are equivalent:*

- (i) *$(X, \sigma(X, X'))$ is metrizable;*
- (ii) *$(X, \sigma(X, X'))$ is Ascoli;*
- (iii) *the topology τ of X coincides with the weak topology $\sigma(X, X')$;*
- (iv) *X' has at most countable Hamel base.*

For the weak* dual space of a Banach space we prove the following result.

Theorem 1.9. *The weak* dual space of a Banach space E is Ascoli if and only if E is finite-dimensional.*

Theorems 1.6 and 1.9 are proved in the last section.

2. A CHARACTERIZATION OF ASCOLI SPACES

For Tychonoff spaces X and Y we denote by $C(X, Y)$ the space of all continuous functions from X to Y . Let \mathfrak{T} be a *directed family* of subsets of X (i.e., if $A, B \in \mathfrak{T}$ then there is $C \in \mathfrak{T}$ such that $A \cup B \subseteq C$) containing all finite subsets. Denote by $\tau_{\mathfrak{T}}$ the set-open topology on $C(X, Y)$ generated by \mathfrak{T} . The subbase of $\tau_{\mathfrak{T}}$ consists of the sets

$$[A; U] = \{f \in C(X, Y) : f(A) \subseteq U\},$$

where $A \in \mathfrak{T}$ and U is an open subset of Y . The space $C(X, Y)$ with the topology $\tau_{\mathfrak{T}}$ is denoted by $C_{\mathfrak{T}}(X, Y)$. If \mathfrak{T} is the family of all finite subsets or the family of all compact subsets of X we obtain the pointwise topology τ_p and the compact-open topology τ_k , respectively, and write $C_p(X, Y)$ and $C_k(X, Y)$. Clearly, $\tau_p \leq \tau_{\mathfrak{T}}$. Denote by

$$\psi : X \times C_{\mathfrak{T}}(X, Y) \rightarrow Y, \quad \psi(x, f) := f(x),$$

the evaluation map.

Definition 2.1. A subset \mathcal{K} of $C(X, Y)$ is called $\tau_{\mathfrak{T}}$ -*evenly continuous* or *evenly continuous* in $C_{\mathfrak{T}}(X, Y)$ if the restriction of the evaluation map ψ onto $X \times \mathcal{K}$ is jointly continuous, i.e., for any $x \in X$, each $f \in \mathcal{K}$ and every neighborhood $O_{f(x)}$ of $f(x)$ there exists a $\tau_{\mathfrak{T}}$ -neighborhood $U_f \subseteq \mathcal{K}$ of f and a neighborhood $O_x \subseteq X$ of x such that

$$\{g(y) : g \in U_f, y \in O_x\} \subseteq O_{f(x)}.$$

So the notion of $\tau_{\mathfrak{T}}$ -even continuity depends on the topology $\tau_{\mathfrak{T}}$ on $C(X, Y)$. Recall that a subset Z of $C(X, Y)$ is called *evenly continuous* if for each $x \in X$, each $y \in Y$ and each neighborhood O_y of y there is a neighborhood O_x of x and a neighborhood V_y of y such that $f(O_x) \subseteq O_y$ whenever $f(x) \in V_y$. Lemma 3.4.18 of [4] states that every evenly continuous subset of $C(X, Y)$ is τ_p -evenly continuous. Since $\tau_p \leq \tau_k$, every τ_p -evenly continuous subset of $C(X, Y)$ is also τ_k -evenly continuous.

If $Y = \mathbb{R}$ we write simply $C(X)$, $C_{\mathfrak{T}}(X)$, $C_p(X)$ or $C_k(X)$, respectively. Then the family

$$\{[A, \varepsilon] : A \in \mathfrak{T}, \varepsilon > 0\}, \quad \text{where } [A, \varepsilon] := \{f \in C(X) : f(A) \subseteq (-\varepsilon, \varepsilon)\},$$

is a base of the topology $\tau_{\mathfrak{T}}$. Recall that a subset Z of $C(X)$ is called *equicontinuous at a point* $x \in X$ if for every $\varepsilon > 0$ there is a neighborhood O_x of x such that $|f(y) - f(x)| < \varepsilon$ for every $y \in O_x$ and each $f \in Z$; Z is *equicontinuous* if it is equicontinuous at each point $x \in X$. It is clear that every equicontinuous subset of $C(X)$ is evenly continuous. This fact and the discussion in the previous paragraph imply the following diagram

$$\text{equicontinuity} \implies \text{even continuity} \implies \begin{matrix} \tau_p\text{-even} \\ \text{continuity} \end{matrix} \implies \begin{matrix} \tau_k\text{-even} \\ \text{continuity} \end{matrix}.$$

It can be shown that none of these implications is reversible.

We defined in [3] a Tychonoff space X to be *Ascoli* if every compact subset $\mathcal{K} \in C_k(X)$ is τ_k -evenly continuous. However in this definition the τ_k -even continuity can be replaced by any other type of even continuity or equicontinuity as item (iv) of the following lemma shows.

Lemma 2.2. *Let X be a Tychonoff space and let \mathfrak{T} and \mathfrak{T}' be directed families of subsets of X containing all finite subsets of X . Then:*

- (i) *every equicontinuous subset \mathcal{K} of $C(X)$ is $\tau_{\mathfrak{T}}$ -evenly continuous;*
- (ii) *every $\tau_{\mathfrak{T}}$ -evenly continuous and $\tau_{\mathfrak{T}}$ -compact subset \mathcal{K} of $C(X)$ is equicontinuous;*
- (iii) *if $\mathfrak{T} \leq \mathfrak{T}'$, then every $\tau_{\mathfrak{T}}$ -evenly continuous subset of $C(X)$ is $\tau_{\mathfrak{T}'}$ -evenly continuous;*
- (iv) *a compact subset \mathcal{K} of $C_k(X)$ is evenly continuous if and only if \mathcal{K} is τ_p -evenly continuous if and only if \mathcal{K} is τ_k -evenly continuous if and only if \mathcal{K} is equicontinuous.*

Proof. (i) Fix $(x, f) \in X \times \mathcal{K}$ and $\varepsilon > 0$. Choose a neighborhood O_x of x such that

$$|g(y) - g(x)| < \varepsilon/2, \quad \forall y \in O_x, \forall g \in \mathcal{K}.$$

Set $A := \{x\}$ and $U_f := (f + [A, \varepsilon/2]) \cap \mathcal{K}$. Now if $h \in U_f$ and $y \in O_x$, we obtain

$$|\psi(y, h) - \psi(x, f)| = |h(y) - f(x)| \leq |h(y) - h(x)| + |h(x) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus ψ is continuous at (x, f) , and hence \mathcal{K} is $\tau_{\mathcal{K}}$ -evenly continuous.

(ii) Let $x \in X$. Fix $\varepsilon > 0$. Since \mathcal{K} is $\tau_{\mathcal{K}}$ -evenly continuous, for every $f \in \mathcal{K}$ and $\varepsilon/2$ -neighborhood $O_{f(x)}$ of $f(x)$ there exists an open $\tau_{\mathcal{K}}$ -neighborhood $U_f \subseteq \mathcal{K}$ of f and a neighborhood $O_{x,f} \subseteq X$ of x such that

$$(2.1) \quad |g(y) - f(x)| < \varepsilon/2, \quad \forall y \in O_{x,f}, \forall g \in U_f.$$

Since \mathcal{K} is $\tau_{\mathcal{K}}$ -compact, there are $f_1, \dots, f_n \in \mathcal{K}$ such that $\mathcal{K} = \bigcup_{i \leq n} U_{f_i}$. Set $O_x := \bigcap_{i \leq n} O_{x,f_i}$. Now, let $y \in O_x$ and $g \in \mathcal{K}$. Choose $1 \leq i \leq n$ such that $g \in U_{f_i}$. Since $y \in O_x \subseteq O_{x,f_i}$ the inequality (2.1) implies

$$|g(y) - g(x)| \leq |g(y) - f_i(x)| + |f_i(x) - g(x)| < \varepsilon.$$

So \mathcal{K} is equicontinuous at x . Since x is arbitrary, \mathcal{K} is equicontinuous.

(iii) is trivial, and (iv) follows from the diagram before the lemma and (ii). \square

So a Tychonoff space X is Ascoli if and only if every compact subset of $C_k(X)$ is equicontinuous.

Recall that a subset A of $C(X)$ is called *pointwise bounded* if the set $\{f(x) : f \in A\}$ has compact closure in \mathbb{R} for every $x \in X$. We shall use the following fact proved in the “if” part of the Ascoli theorem [4, Theorem 3.4.20].

Proposition 2.3. *Let X be a Tychonoff space and \mathcal{K} be an evenly continuous pointwise bounded subset of $C(X)$. Then the τ_p -closure \bar{A} of A is τ_k -compact and evenly continuous. Moreover, $\tau_k|_{\bar{A}} = \tau_p|_{\bar{A}}$.*

Recall that the *free locally convex space* $L(X)$ over a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $i : X \rightarrow L(X)$ such that every continuous mapping f from X to a locally convex space E gives rise to a unique continuous linear operator $\bar{f} : L(X) \rightarrow E$ with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding [5, 22, 26].

The space $L(X)$ admits a canonical linear monomorphism $\Delta_X : L(X) \rightarrow C_k(C_k(X))$, extending the evaluation map δ_X , defined by the assignment

$$\Delta_X(a_1x_1 + \dots + a_nx_n)(f) := a_1f(x_1) + \dots + a_nf(x_n),$$

where $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, $a_1, \dots, a_n \in \mathbb{R}$ and $f \in C_k(X)$.

In [3] we show that a space X is Ascoli if and only if the evaluation map $\delta_X : X \mapsto C_k(C_k(X))$, $\delta_X(x)(f) := f(x)$, is a topological embedding. So Theorem 1.1 gives a “locally convex linear” characterization of Ascoli spaces. The closure of a subset A of a topological space (X, τ) we denote by \bar{A}^τ . Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Necessity. In [22] Raikov showed that the topology ν_X of $L(X)$ is the topology of uniform convergence on equicontinuous pointwise bounded subsets A of $C(X)$. Lemma 2.2 and Proposition 2.3 imply that the closure \bar{A}^{τ_k} of A is τ_k -compact and τ_k -evenly continuous. Conversely, if A is τ_k -evenly continuous and τ_k -compact, then A is equicontinuous by Lemma 2.2. Clearly, A is τ_p -compact, and hence $\{f(x) : f \in A\}$ is compact in \mathbb{R} for every $x \in X$. Therefore A is also pointwise bounded. Hence the topology ν_X coincides with the topology of uniform convergence on compact evenly continuous subsets of $C_k(X)$. Since the space X is Ascoli, every compact subset of $C_k(X)$ is evenly continuous. Taking into account that the canonical map is injective we obtain

that ν_X coincides with the compact-open topology inherited from $C_k(C_k(X))$. Thus Δ_X is an embedding.

Sufficiency. If Δ_X is an embedding, then the canonical map $\delta_X = \Delta_X|_X$ is an embedding as well (recall that X is a subspace of $L(X)$). Thus X is an Ascoli space by [3, Proposition 5.4]. \square

Below we give an application of Theorem 1.1. First we recall some definitions.

Following Markov [16], a topological group $A(X)$ is called *the (Markov) free abelian topological group* over X if $A(X)$ satisfies the following conditions: (i) there is a continuous mapping $i : X \rightarrow A(X)$ such that $i(X)$ algebraically generates $A(X)$, and (ii) if $f : X \rightarrow G$ is a continuous mapping to an abelian topological group G , then there exists a continuous homomorphism $\bar{f} : A(X) \rightarrow G$ such that $f = \bar{f} \circ i$. The topological group $A(X)$ always exists and is essentially unique, the mapping i is a topological embedding [16]. Note also that the identity map $id_X : X \rightarrow X$ extends to a canonical homomorphism $id_{A(X)} : A(X) \rightarrow L(X)$ which is an embedding of topological groups [25, 27].

Following Michael [17], a topological space X is called an \aleph_0 -space, if X is a regular space with a countable k -network (a family \mathcal{N} of subsets of X is called a k -network in X if, whenever $K \subseteq U$ with K compact and U open in X , then $K \subseteq \bigcup \mathcal{F} \subseteq U$ for some finite family $\mathcal{F} \subseteq \mathcal{N}$). A topological space X is called *cosmic* [17], if X is a regular space with a countable network (a family \mathcal{N} of subsets of X is called a *network* in X if, whenever $x \in U$ with U open in X , then $x \in N \subseteq U$ for some $N \in \mathcal{N}$). Following Banach [1], a topological space X is called a \mathfrak{P}_0 -space if X has a countable Pytkeev network (a family \mathcal{N} of subsets of a topological space X is called a *Pytkeev network* if \mathcal{N} is a network in X and for every point $x \in X$ and every open set $U \subseteq X$ and a set A accumulating at x there is a set $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite). Any \mathfrak{P}_0 -space is an \aleph_0 -space, see [1].

It is known (see [12]) that, for a Tychonoff space X , the space $L(X)$ is cosmic if and only if the group $A(X)$ is cosmic if and only if the space X is cosmic. For k -spaces the next corollary is proved in [7].

Corollary 2.4. *Let X be an Ascoli space. Then:*

- (i) $L(X)$ is an \aleph_0 -space if and only if $A(X)$ is an \aleph_0 -space if and only if X is an \aleph_0 -space;
- (ii) $L(X)$ is a \mathfrak{P}_0 -space if and only if $A(X)$ is a \mathfrak{P}_0 -space if and only if X is a \mathfrak{P}_0 -space.

Proof. If $L(X)$ is an \aleph_0 -space (a \mathfrak{P}_0 -space), then so is $A(X)$ as a subspace of $L(X)$. Analogously, if $A(X)$ is an \aleph_0 -space (a \mathfrak{P}_0 -space), then so is X as a subspace of $A(X)$. If X is an \aleph_0 -space (a \mathfrak{P}_0 -space), then so are $C_k(X)$ and $C_k(C_k(X))$ by [17] (respectively, [1]). As X is an Ascoli space, $L(X)$ is a subspace of $C_k(C_k(X))$ by Theorem 1.1. So $L(X)$ is an \aleph_0 -space (respectively, a \mathfrak{P}_0 -space). \square

We do not know whether the condition on X to be an Ascoli space can be omitted as for cosmic spaces in [12], namely: Does there exist a non-Ascoli \aleph_0 -space (a non-Ascoli \mathfrak{P}_0 -space) X such that $L(X)$ is an \aleph_0 -space (respectively, a \mathfrak{P}_0 -space)?

3. THE ASCOLI PROPERTY FOR DIRECT SUMS OF LOCALLY CONVEX SPACES

Let us recall some basic notions used in what follows. We denote by e the unit of a group G .

For a non-empty family $\{G_i\}_{i \in I}$ of groups, the *direct sum* of G_i is denoted by

$$\bigoplus_{i \in I} G_i := \left\{ (g_i)_{i \in I} \in \prod_{i \in I} G_i : g_i = e_i \text{ for almost all } i \right\},$$

and we denote by j_k the natural embedding of G_k into $\bigoplus_{i \in I} G_i$; that is,

$$j_k(g) = (g_i) \in \bigoplus_{i \in I} G_i, \text{ where } g_i = g \text{ if } i = k \text{ and } g_i = e_i \text{ if } i \neq k.$$

If $\{G_i\}_{i \in I}$ is a non-empty family of topological groups the *final group topology* \mathcal{T}_f on $\bigoplus_{i \in I} G_i$ with respect to the family of canonical homomorphisms $j_k : G_k \rightarrow \bigoplus_{i \in I} G_i$ is the finest group topology on $\bigoplus_{i \in I} G_i$ such that all j_k are continuous. For the sake of simplicity we shall identify an element $g_k \in G_k$ with its image $j_k(g_k)$ in $\bigoplus_{i \in I} G_i$ and write simply g_k .

For an element $v = g_{i_1} + \dots + g_{i_n} \in \bigoplus_{i \in I} G_i$ with $g_{i_k} \neq e_{i_k}$ for every $1 \leq k \leq n$, we set $\text{supp}(v) := \{i_1, \dots, i_n\}$. The *support* of a subset A of $\bigoplus_{i \in I} G_i$ is the set

$$\text{supp}(A) := \bigcup_{v \in A} \text{supp}(v).$$

Let $\{(G_i, \tau_i)\}_{i \in I}$ be a non-empty family of topological groups and $\mathcal{N}(G_i)$ a basis of open neighborhoods at the identity in G_i , for each $i \in I$. For each $i \in I$, fix $U_i \in \mathcal{N}(G_i)$ and put

$$\prod_{i \in I} U_i := \left\{ (g_i)_{i \in I} \in \prod_{i \in I} G_i : g_i \in U_i \text{ for all } i \in I \right\}.$$

Then the sets of the form $\prod_{i \in I} U_i$, where $U_i \in \mathcal{N}(G_i)$ for every $i \in I$, form a basis of open neighborhoods at the identity of a topological group topology \mathcal{T}_b on $\prod_{i \in I} G_i$ that is called the *box topology*. Clearly, $\mathcal{T}_b \leq \mathcal{T}_f$ on $\bigoplus_{i \in I} G_i$. It is well-known that if I is countable, then $\mathcal{T}_b = \mathcal{T}_f$.

Recall that a subset A of a topological space X is called *functionally bounded* in X if every continuous function on X is bounded on A .

Proposition 3.1. *Let $\{(G_i, \tau_i)\}_{i \in I}$ be a non-empty family of topological groups and τ be a group topology on $\bigoplus_{i \in I} G_i$ such that $\mathcal{T}_b \leq \tau \leq \mathcal{T}_f$. If A is a functionally bounded subset of $(\bigoplus_{i \in I} G_i, \tau)$, then $\text{supp}(A)$ is finite.*

Proof. Suppose for a contradiction that $\text{supp}(A)$ is infinite. Take a one-to-one sequence $\{i_n\}_{n \in \mathbb{N}}$ in $\text{supp}(A)$. Then the projection of τ on $\bigoplus_{n \in \mathbb{N}} G_{i_n}$ is \mathcal{T}_b . Clearly, the image B of A in the group $G := (\bigoplus_{n \in \mathbb{N}} G_{i_n}, \mathcal{T}_b)$ is also functionally bounded and $\text{supp}(B) = \mathbb{N}$. Choose a sequence $\{b_k = (g_{i_n}^k) : k \in \mathbb{N}\}$ in B such that for every $k \in \mathbb{N}$ there is an index $i_{n_{k+1}} \in \text{supp}(b_{k+1})$ such that

$$(3.1) \quad i_{n_{k+1}} \notin \bigcup_{j \leq k} \text{supp}(b_j).$$

Clearly, the sequence $\{b_k\}$ is also functionally bounded in G . For every $k \in \mathbb{N}$, take a symmetric neighborhood $U_{i_{n_k}}$ of the identity in $G_{i_{n_k}}$ such that $g_{i_{n_k}}^k \notin U_{i_{n_k}}$. If $i_n \notin \{i_{n_1}, i_{n_2}, \dots\}$, we set $U_{i_n} = G_{i_n}$. Set $U := G \cap \prod_{n \in \mathbb{N}} U_{i_n} \in \mathcal{T}_b$. Now, if $k < m$, then $b_k U \cap b_m U = \emptyset$ since, otherwise, for some $h, t \in U_{i_{n_m}}$ we would have

$$g_{i_{n_m}}^m t = g_{i_{n_m}}^k h = h \text{ by (3.1), and hence } g_{i_{n_m}}^m = h t^{-1} \in U_{i_{n_m}} U_{i_{n_m}}$$

that contradicts the choice of $g_{i_{n_m}}^m$ and $U_{i_{n_m}}$. So, if $V \in \mathcal{T}_b$ is such that $\overline{VV} \subseteq U$, then $\{b_k V\}_{k \in \mathbb{N}}$ is a discrete family in G . Thus the sequence $\{b_k\}$ is not functionally bounded in G , a contradiction. \square

Let κ be an infinite cardinal, $\mathbb{V}_\kappa = \bigoplus_{i \in \kappa} \mathbb{R}_i$ be a vector space of dimension κ over \mathbb{R} , τ_κ be the box topology on \mathbb{V}_κ , μ_κ and ν_κ be the maximal and maximal locally convex vector topologies on \mathbb{V}_κ respectively. Clearly, $\tau_\kappa \subseteq \nu_\kappa \subseteq \mu_\kappa$ and $L(D) \cong (\mathbb{V}_\kappa, \nu_\kappa)$, where D is a discrete space of cardinality κ . It is well-known that $\tau_\omega = \nu_\omega = \mu_\omega$ (see [14, Proposition 4.1.4]). However, if κ is uncountable, then (see [21] or [6, Theorem 2.1])

$$\tau_\kappa \subsetneq \nu_\kappa \subsetneq \mu_\kappa.$$

We shall use the following simple description of the topology μ_κ given in the proof of Theorem 1 in [21]. For each $i \in \kappa$, choose some $\lambda_i \in \mathbb{R}_i^+$, $\lambda_i > 0$, and denote by \mathcal{S}_κ the family of all subsets

of \mathbb{V}_κ of the form

$$\bigcup_{i < \kappa} \left([-\lambda_i, \lambda_i] \times \prod_{j < \kappa, j \neq i} \{0\} \right).$$

For every sequence $\{S_k\}_{k \in \omega}$ in \mathcal{S}_κ , we put

$$\sum_{k \in \omega} S_k := \bigcup_{k \in \omega} (S_0 + S_1 + \cdots + S_k),$$

and denote by \mathcal{N}_κ the family of all subsets of \mathbb{V}_κ of the form $\sum_{k \in \omega} S_k$. It is easy to check that \mathcal{N}_κ is a base at zero $\mathbf{0}$ for μ_κ and the family $\widehat{\mathcal{N}}_\kappa := \{\text{conv}(V) : V \in \mathcal{N}_\kappa\}$ of convex hulls is a base at $\mathbf{0}$ for ν_κ (see [21]). For $(x_i) \in \mathbb{V}_\kappa$, we denote $\text{supp}(x_i) := \{i \in \kappa : x_i \neq 0\}$.

We shall use also the following proposition to show that a space is not Ascoli.

Proposition 3.2 ([10]). *Assume X admits a family $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of X , a subset $A = \{a_i : i \in I\} \subset X$ and a point $z \in X$ such that*

- (i) $a_i \in U_i$ for every $i \in I$;
- (ii) $|\{i \in I : C \cap U_i \neq \emptyset\}| < \infty$ for each compact subset C of X ;
- (iii) z is a cluster point of A .

Then X is not an Ascoli space.

Theorem 3.3. *Let κ be an uncountable cardinal and let τ be a vector topology on \mathbb{V}_κ such that $\tau_\kappa \subseteq \tau \subseteq \nu_\kappa$. Then $(\mathbb{V}_\kappa, \tau)$ does not have the Ascoli property. In particular, $L(D)$ is not Ascoli for every uncountable discrete space D .*

Proof. For every $n \in \mathbb{N}$, set

$$R_n = \left\{ \mathbf{x} = (x_i) \in \mathbb{V}_\kappa : |\text{supp}(\mathbf{x})| = n \text{ and } x_i = \frac{1}{n^2} \text{ for every } i \in \text{supp}(\mathbf{x}) \right\}$$

and $R = \bigcup_{n \in \mathbb{N}} R_n$. Clearly, $\mathbf{0} \notin R$.

We claim that $\mathbf{0} \in \overline{R}^{\nu_\kappa}$ and hence $\mathbf{0} \in \overline{R}^\tau$. Take arbitrarily an open convex neighborhood W of $\mathbf{0}$ in ν_κ . Choose a neighborhood $\sum_{k \in \omega} S_k$ of $\mathbf{0}$ in μ_κ such that $\sum_{k \in \omega} S_k \subseteq W$. Since κ is uncountable, there is a positive number $c > 0$ and an uncountable set J of indices such that $\lambda_j^0 > c$ for all $j \in J$, where the positive numbers λ_j^0 define S_0 . Take $n \in \mathbb{N}$ with $1/n < c$ and a finite subset $J_0 = \{j_1, \dots, j_n\}$ of J . For every $1 \leq l \leq n$ we set

$$\mathbf{x}_l = (x_i^l)_{i \in \kappa}, \quad \text{where } x_i^l = \frac{1}{n} \text{ if } i = j_l, \text{ and } x_i^l = 0 \text{ otherwise.}$$

So $\mathbf{x}_l \in S_0 \subset \sum_{n \in \omega} S_n \subseteq W$ for every $1 \leq l \leq n$. Since W is convex the element

$$\mathbf{x} := \frac{1}{n}(\mathbf{x}_1 + \cdots + \mathbf{x}_n)$$

belongs to W . By construction, $\mathbf{x} \in R_n$. Thus $\mathbf{0} \in \overline{R}^{\nu_\kappa}$ and the claim is proven.

For every $n \in \mathbb{N}$, set

$$W_n := \mathbb{V}_\kappa \cap \prod_{i < \kappa} \left(-\frac{1}{10n^3}, \frac{1}{10n^3} \right) \in \tau_\kappa \subseteq \tau.$$

Now for every $\mathbf{x} \in R_n$, set $U(\mathbf{x}) := \mathbf{x} + W_n$. Since

$$(3.2) \quad \frac{1}{n^2} - \frac{1}{10n^3} > \frac{1}{(n+1)^2} + \frac{1}{10(n+1)^3} > 0,$$

we note that $U(\mathbf{x}) \cap U(\mathbf{y}) = \emptyset$ for every distinct $\mathbf{x}, \mathbf{y} \in R$.

To prove that the space $(\mathbb{V}_\kappa, \nu_\kappa)$ is not Ascoli it is enough to show that the families R and $\{U(\mathbf{x}) : \mathbf{x} \in R\}$ satisfy conditions (i)–(iii) of Proposition 3.2 with $z = \mathbf{0}$. Clearly, (i) holds and (iii) is true by the claim. Let us check (ii).

Let K be a compact subset of $(\mathbb{V}_\kappa, \tau)$. Then, by Proposition 3.1, there is a finite subfamily $F = \{i_1, \dots, i_n\}$ of κ such that $\text{supp}(K) \subseteq F$. If $\text{supp}(\mathbf{x}) \not\subseteq F$, (3.2) implies that $U(\mathbf{x}) \cap K = \emptyset$. On the other hand, it is easily seen that the number of $\mathbf{x} \in R$ such that $\text{supp}(\mathbf{x}) \subseteq F$ is finite. This means that (ii) of Proposition 3.2 also holds true. Thus $(\mathbb{V}_\kappa, \tau)$ is not Ascoli. \square

Proof of Theorem 1.2. For each $i < \kappa$, by [15, §20.5(5)], represent E_i in the form $E_i = \mathbb{R} \oplus \tilde{E}_i$, where \tilde{E}_i is a closed subspace of E_i . Then

$$(E, \tau) = (\mathbb{V}_\kappa, \tau|_{\mathbb{V}_\kappa}) \times (\tilde{E}, \tau|_{\tilde{E}}), \quad \text{where } \tilde{E} := \bigoplus_{i \in \kappa} \tilde{E}_i.$$

Since the space $(\mathbb{V}_\kappa, \tau|_{\mathbb{V}_\kappa})$ has uncountable tightness by Theorem 2.1 of [6], so is (E, τ) . As the direct summand $(\mathbb{V}_\kappa, \tau|_{\mathbb{V}_\kappa})$ of (E, τ) is not Ascoli by Theorem 3.3, the space (E, τ) is not an Ascoli space by Proposition 5.2 of [3]. \square

4. PROOF OF THEOREM 1.3

Let $\{(X_n, \tau_n)\}_{n \in \omega}$ be a sequence of topological spaces such that $X_n \subseteq X_{n+1}$ and $\tau_{n+1}|_{X_n} = \tau_n$ for all $n \in \omega$. The union $X = \bigcup_{n \in \omega} X_n$ with the *weak topology* τ (i.e., $U \in \tau$ if and only if $U \cap X_n \in \tau_n$ for every $n \in \omega$) is called the *inductive limit* of the sequence $\{(X_n, \tau_n)\}_{n \in \omega}$ and it is denoted by $(X, \tau) = \varinjlim (X_n, \tau_n)$. If X_n is closed in X_{n+1} for every $n \in \omega$, then, clearly, X_n is a closed subspace of X . A topological space X is called a k_ω -space (an \mathcal{MK}_ω -space) if it is the inductive limit of an increasing sequence $\{C_n\}_{n \in \omega}$ of its (respectively, metrizable) compact subsets. In [24, Lemma 9.3] Steenrod proved the following useful result.

Proposition 4.1. *If K is a compact subset of $(X, \tau) = \varinjlim (X_n, \tau_n)$, then $K \subseteq X_n$ for some $n \in \mathbb{N}$.*

Corollary 4.2. *If $(X, \tau) = \varinjlim (X_n, \tau_n)$ is a μ -space and A is a functionally bounded subset of X , then $A \subseteq X_n$ for some $n \in \omega$.*

Proof. By definition, X is a μ -space if and only if the closure \overline{A} of any functionally bounded subset A of X is compact. So Proposition 4.1 applied. \square

In what follows we shall omit τ_n and write simply $X = \varinjlim X_n$.

The following proposition plays a crucial role in the proof of Theorem 1.3.

Proposition 4.3. *Let $X = \varinjlim X_n$ be the inductive limit of a sequence $\{X_n\}_{n \in \omega}$ of metrizable groups such that X_n is a closed non-open subgroup of X_{n+1} for every $n \in \omega$. If X is an Ascoli space, then all the X_n are locally compact.*

Proof. Suppose for a contradiction that there is X_i , say X_0 , which is not locally compact. For every $i \in \omega$ we denote by ρ_i a left invariant metric on X_i and set

$$B_{n,i} := \{x \in X_i : \rho_i(x, e) < 2^{-n}\}, \quad n \in \omega.$$

Step 1. Consider the open base of neighborhoods $\{B_{n,0}\}_{n \in \omega}$ of the unit e of X_0 , so $\overline{B_{n+1,0}} \subseteq B_{n,0}$. Then there is a strictly increasing sequence $\{n_k\}_{k \in \omega}$ such that $n_{k+1} > n_k + 1$ and for every $k \in \omega$, the set $\overline{B_{n_k,0}} \setminus B_{n_{k+1,0}}$ is not compact. (Indeed, otherwise, there is n_0 such that $\overline{B_{n_0,0}} \setminus B_{n_{n_0+1,0}}$ is compact for all $n \geq n_0$. Since $B_{n,0}$ converges to e , we obtain that $\overline{B_{n_0,0}}$ is compact. So X_0 is locally compact, a contradiction.)

Set $P_k := \overline{B_{n_k,0}} \setminus B_{n_k+1,0}$. Then P_k is metrizable and non-compact, and hence P_k is not pseudocompact. By [4, Theorem 3.10.22] there exists a locally finite collection $\{W_{n,k}\}_{n < \omega}$ of nonempty open subsets of P_k . We may assume in addition that every $W_{n,k} \subseteq \text{Int}(P_k)$. Note that the family

$$\mathcal{W}_m := \{W_{n,i} : n \in \omega, i \leq m\}$$

is also locally finite in X_0 for every $m \in \omega$. For every $n, k \in \omega$ pick arbitrarily $x_{n,k} \in W_{n,k}$.

Step 2. We claim that for every $k \geq 1$ there are

- (a) a one-to-one sequence $\{y_{n,k}\}_{n \in \omega}$ in $X_k \setminus X_{k-1}$ converging to the unit $e \in X$;
- (b) for every $n \in \omega$, an open neighborhood $U_{n,k}$ of $a_{n,k} := x_{n,k-1}y_{n,k}$ in X ;

such that

- (c) $U_{n,k} \cap X_{k-1} = \emptyset$ for every $n \in \omega$;
- (d) the family

$$\mathcal{V}_k := \{U_{n,i} \cap X_k : n \in \omega, 1 \leq i \leq k\}$$

is locally finite in X_k .

Indeed, for every $k \geq 1$, let $\{y_{n,k}\}_{n \in \omega}$ be an arbitrary one-to-one sequence in $X_k \setminus X_{k-1}$ converging to e (such a sequence exists because X_{k-1} is not open in X_k by assumption). For every $k \geq 1$ and each $n \in \omega$ choose an open symmetric neighborhood $V_{n,k}$ of e in X such that

- (α) $V_{n,k}^3 \cap X_i \subseteq B_{n,i}$ for every $0 \leq i \leq n$;
- (β) $(y_{n,k} \cdot V_{n,k}^3) \cap X_{k-1} = \emptyset$ (recall that X_{k-1} is closed in X).

For every $k \geq 1$ and each $n \in \omega$ set

$$a_{n,k} := x_{n,k-1}y_{n,k} \quad \text{and} \quad U_{n,k} := a_{n,k}V_{n,k}.$$

Clearly, (a) and (b) hold. Also (c) holds since if $U_{n,k} \cap X_{k-1} \neq \emptyset$ for some $k \geq 1$ and $n \in \omega$, then $x_{n,k-1}y_{n,k}v \in X_{k-1}$ for some $v \in V_{n,k}$. So $y_{n,k}v \in x_{n,k-1}^{-1}X_{k-1} = X_{k-1}$ that contradicts (β).

Let us check that \mathcal{V}_k is locally finite in X_k for every $k \geq 1$. Fix $x \in X_k$ and consider the two possible cases.

Case 1. Let $x \in X_k \setminus X_0$. So $\rho_k(x, X_0) > 0$ as X_0 is closed in X_k . For every $1 \leq i \leq k$, since $y_{n,i} \rightarrow e$ in X and $x_{n,i-1} \in X_0$, the condition (α) implies (note that $V_{n,i} \cap X_k \subseteq B_{n,k}$ for every $1 \leq i \leq k < n$)

$$\lim_n \rho_k(U_{n,i} \cap X_k, X_0) = \lim_n \rho_k(y_{n,i}V_{n,i} \cap X_k, X_0) \leq \lim_n \rho_k(y_{n,i}B_{n,k}, e) = 0.$$

Hence there is an open neighborhood U_x of x in X such that $U_x \cap X_k$ intersects only with a finite subfamily of \mathcal{V}_k .

Case 2. Let $x \in X_0$. Choose an open symmetric neighborhood U_x of e in X such that $xU_x^3 \cap X_0$ intersects only with a finite subfamily of \mathcal{W}_k . We claim that $xU_x \cap X_k$ intersects only a finite subfamily of \mathcal{V}_k . Indeed, assuming the converse we can find $1 \leq i \leq k$ such that $(xU_x \cap X_k) \cap U_{n,i} \neq \emptyset$ for every $n \in I$, where I is an infinite subset of ω . Then for every $n \in I$ there are $u_n \in X_k$, $t_n \in U_x$ and $z_n \in V_{n,i}$ such that

$$u_n = x \cdot t_n = x_{n,i-1}y_{n,i}z_n.$$

Note that $z_n = y_{n,i}^{-1}x_{n,i-1}^{-1}u_n \in V_{n,i} \cap X_k$ belongs to $U_x \cap X_k$ for all sufficiently large $n \in I$ by (α), and also $y_{n,i} \in U_x \cap X_k$ for all sufficiently large $n \in I$ because $y_{n,i} \rightarrow e$. So

$$x_{n,i-1} = x \cdot (t_n z_n^{-1} y_{n,i}^{-1}) \in (xU_x^3 \cap X_0) \cap W_{n,i-1}$$

for all sufficiently large $n \in I$. But this contradicts the choice of U_x .

Cases 1 and 2 show that \mathcal{V}_k is locally finite in X_k .

Step 3. Let us show that the families

$$\mathcal{A} := \{a_{i,k} : i \in \omega, k \geq 1\}, \quad \mathcal{U} := \{U_{i,k} : i \in \omega, k \geq 1\}$$

and $z := e$ satisfy (i)-(iii) of Proposition 3.2. Indeed, (i) is clear. To check (ii) let K be a compact subset of X . By Proposition 4.1, there is $m \in \omega$ such that $K \subseteq X_m$. So (c) implies that if $K \cap U_{n,i} \neq \emptyset$, then $i \leq m$, and hence $U_{n,i} \cap X_m \in \mathcal{V}_m$. Since the family \mathcal{V}_m is locally finite in X_m , we obtain that K intersects only a finite subfamily of \mathcal{U} that proves (ii).

To prove (iii) let V be an open neighborhood of e . Take an open neighborhood U of e such that $U^2 \subseteq V$, and choose $k_0 \in \omega$ such that $W_{i,k_0} \subseteq X_0 \cap U$ for every $i \in \omega$. So $x_{i,k_0} \in U$ for every $i \in \omega$. Since $\lim_i y_{i,k_0+1} = e$ we obtain that $a_{i,k_0+1} = x_{i,k_0} y_{i,k_0+1} \in U \cdot U \subseteq V$ for all sufficiently large i . Thus $e \in \overline{A}$ and (iii) holds. Finally, Proposition 3.2 implies that the group X is not Ascoli which is a desired contradiction. \square

The following theorem is a more precise version of Theorem 1.3.

Theorem 4.4. *Let $X = \varinjlim X_n$ be the inductive limit of a sequence $\{X_n\}_{n \in \omega}$ of metrizable groups such that X_n is a closed subgroup of X_{n+1} for every $n \in \omega$. Then the following assertions are equivalent:*

- (i) X is an Ascoli space;
- (ii) one of the following assertions holds:
 - (ii)₁ there is $m \in \omega$ such that X_n is open in X_{n+1} for every $n \geq m$, so X is metrizable;
 - (ii)₂ all the X_n are locally compact, so X contains an open \mathcal{MK}_ω -subspace and hence X is sequential.

Proof. (i) \Rightarrow (ii) If there is $m \in \omega$ such that X_n is open in X_{n+1} for every $n \geq m$, then X_m is an open subgroup of X . Thus X is metrizable. Assume that for infinitely many $n \in \omega$ the group X_n is not open in X_{n+1} . Without loss of generality we can assume that X_n is not open in X_{n+1} for all $n \in \omega$. Since X is Ascoli, Proposition 4.3 implies that all the X_n are locally compact. Let Y_n be an open σ -compact subgroup of X_n . As all the Y_n are metrizable, the group $Y := \varinjlim Y_n$ is an \mathcal{MK}_ω -space, and hence Y is sequential. Clearly, Y is an open subgroup of X . Thus X is sequential.

(ii) \Rightarrow (i) follows from the Ascoli theorem [4]. \square

Recall that a locally convex space E is a *strict (LF)-space* if E is the inductive limit of a sequence $\{E_n\}_{n \in \omega}$ of Fréchet (= complete and metrizable locally convex) spaces, see [15]. Now we prove Corollary 1.4.

Proof of Corollary 1.4. For every $n \in \omega$ the space E_n is closed in E_{n+1} as a complete subspace of a complete metrizable space. Taking into account that a locally convex space E is locally compact if and only if E is finite dimensional, the assertion follows from Theorem 4.4. \square

It is mentioned in [20, Footnote 2] that van Douwen has shown the following: if even one of the factors in the direct sum $X = (\bigoplus_{n \in \omega} X_n, \mathcal{T}_b)$ of a sequence $\{X_n\}_{n \in \omega}$ of metrizable groups with the box topology \mathcal{T}_b is not locally compact and infinitely many of the X_n are not discrete, then X is not sequential. The next corollary generalizes this result.

Corollary 4.5. *Let $\{X_n\}_{n \in \omega}$ be a sequence of metrizable groups such that infinitely many of the X_n are not discrete and let $X = (\bigoplus_{n \in \omega} X_n, \mathcal{T}_b)$ be the direct sum endowed with the box topology \mathcal{T}_b . Then the following assertions are equivalent:*

- (i) all the X_n are locally compact;
- (ii) X is sequential;
- (iii) X is an Ascoli space.

Proof. Set $\tilde{X}_n := \bigoplus_{i \leq n} X_i$, $n \in \omega$. Then $X = \varinjlim \tilde{X}_n$ and Theorem 4.4 applies. \square

Corollary 4.6. *Let $E = \bigoplus_{n \in \omega} E_n$ be the direct locally convex sum of a sequence $\{E_n\}_{n \in \omega}$ of nontrivial metrizable locally convex spaces. Then E is Ascoli if and only if $E = \varphi$.*

Proof. Taking into account that a locally convex space E is locally compact if and only if E is finite dimensional, the assertion follows from Corollary 4.5. \square

In particular, the space $\mathbb{R}^\omega \times \varphi$ is not Ascoli, and hence the product of a metrizable space and a sequential space can be not Ascoli.

5. PROOFS OF THEOREMS 1.6 AND 1.9

Recall that a subset A of a locally convex space E is called *bounded* if for every neighborhood U of zero there is $\lambda > 0$ such that $A \subseteq \lambda U$. If (X, Y) is a dual pair of vector spaces and L is a linear subspace of X we set $L^\perp := \{y : y(x) = 0 \forall x \in L\}$. We denote by X_w the space X endowed with the weak topology $\sigma(X, X')$. Below we prove a more precise version of Theorem 1.6.

Theorem 5.1. *If a c_0 -barrelled space X is weakly Ascoli, then:*

- (i) *every $\sigma(X', X)$ -bounded subset of X' is finite dimensional;*
- (ii) *the topology τ of X coincides with the weak topology;*
- (iii) *X is linearly isomorphic to a dense subspace of \mathbb{R}^Γ , where Γ is a Hamel base of X' .*

Proof. (i) Suppose for a contradiction that there exists an infinite dimensional $\sigma(X', X)$ -bounded subset of X' . Then X' has an independent and $\sigma(X', X)$ -bounded sequence $\{y_n\}_{n \in \omega}$. Clearly, $(1/n)y_n \rightarrow 0$ in $\sigma(X', X)$. So the set $K := \{0\} \cup \{(1/n)y_n\}_{n \in \omega}$ is compact in $\sigma(X', X)$. Let us show that $(1/n^2)y_n \rightarrow 0$ in the compact-open topology. Indeed, fix a standard neighborhood $[C, \varepsilon]$ of zero in $C_k(X_w)$, where C is a $\sigma(X, X')$ -compact subset of X , $\varepsilon > 0$ and

$$[C, \varepsilon] := \{f \in C_k(X_w) : |f(x)| < \varepsilon \text{ for every } x \in C\}.$$

Since X is c_0 -barrelled, K is equicontinuous, and hence the polar K° of K is a neighborhood of zero in X by Theorem 8.6.4 of [18]. As C is weakly compact it is bounded in X . So there is $m > 0$ such that $C \subseteq mK^\circ$, and hence $K \subseteq (1/m)C^\circ \subseteq (2/m\varepsilon)[C, \varepsilon]$. Since $[C, \varepsilon]$ is absolutely convex we obtain

$$\frac{1}{n^2}y_n = \frac{1}{n} \left(\frac{1}{n}y_n \right) \in [C, \varepsilon] \text{ for every } n > \frac{m\varepsilon}{2}.$$

Thus $(1/n^2)y_n \rightarrow 0$ in $C_k(X_w)$. Since X_w is Ascoli to get a contradiction it is sufficient to show that the compact set $\mathcal{K} := \{0\} \cup \{(1/n^2)y_n\}_{n \in \omega} \subseteq C_k(X_w)$ is not equicontinuous at zero, see Lemma 2.2.

Let U be a basic neighborhood of zero $0 \in X_w$. So there are $\delta > 0$ and $z_1, \dots, z_n \in X'$ such that

$$U = \{x \in X : |z_i(x)| < \delta \text{ for } i = 1, \dots, n\}.$$

Denote by L the span of the vectors z_1, \dots, z_n in X' . Then L is a $\sigma(X', X)$ -closed finite dimensional subspace of X' . Hence there is a closed subspace H of $X'_{w*} := (X', \sigma(X', X))$ such that $X'_{w*} = L \oplus H$, see [15, §20.5(5)]. Since L is finite dimensional and the y_n are independent, there is n such that $(1/n^2)y_n \notin L$. As $L = L^{\perp\perp}$ we obtain that $(1/n^2)y_n(x) = a \neq 0$ for some $x \in L^\perp \subseteq X$. Finally, since $(1/a)x \in L^\perp \subseteq U$ and $(1/n^2)y_n((1/a)x) = 1$ we obtain that \mathcal{K} is not equicontinuous at 0. This contradiction shows that every $\sigma(X', X)$ -bounded subset of X' must be finite dimensional.

(ii) To prove that $\tau = \sigma(X, X')$ we have to show that every τ -neighborhood U of zero in X is also a weak neighborhood of zero. Note that τ is the polar topology determined by equicontinuous subsets of X' by [18, Theorem 8.6.6], and every equicontinuous subset of X' is $\sigma(X', X)$ -bounded by [18, Theorem 8.6.5]. So U contains a neighborhood of zero of the form A° , where A is a $\sigma(X', X)$ -bounded subset of X' . We proved in (i) that A is finite dimensional. Hence, as in the previous paragraph, there is a finite dimensional subspace L_A of X' , a $\sigma(X', X)$ -closed subspace H_A of X' and a standard compact neighborhood $W_A = \prod_{1 \leq i \leq \dim(L_A)} [-a_i, a_i]$ of zero in L_A such that

$$(X', \sigma(X', X)) = L_A \oplus H_A \text{ and } A \subseteq W_A \times \{0\}.$$

Set $M := H_A^\perp, G := L_A^\perp$, so $X = M \oplus G$. Put $n := \dim(L_A)$, $B := W_A \times \{0\}$ and define $F = \{z_1, \dots, z_n\} \subseteq X'$ by

$$z_i(m + g) := a_i m_i, \text{ where } m = (m_1, \dots, m_n) \in M \text{ and } g \in G.$$

Then $F^\circ = B^\circ \subseteq A^\circ \subseteq U$. Thus $\tau = \sigma(X, X')$.

(iii) It is well known that for every locally convex space E the space E_w is linearly homeomorphic to a dense subspace of \mathbb{R}^Γ , where Γ is a Hamel base of E' . As $X = X_w$ by (ii), the assertion follows. \square

Proof of Corollary 1.7. Let X be a weakly Ascoli complete c_0 -barrelled space. Then Theorem 1.6 implies that $X = \mathbb{R}^\Gamma$, where Γ is a Hamel base of X' . Therefore X is also a $k_{\mathbb{R}}$ -space by [19]. Conversely, if $X = \mathbb{R}^\Gamma$, then $X = X_w$ is even a barrelled complete $k_{\mathbb{R}}$ -space. The last assertion is clear. \square

Corollary 5.2. *Let (X, τ) be a locally convex space such that its completion $(\overline{X}, \overline{\tau})$ is c_0 -barrelled. If $(X, \sigma(X, X'))$ is Ascoli, then $\tau = \sigma(X, X')$ and $(\overline{X}, \sigma(\overline{X}, X'))$ is an Ascoli space.*

Proof. Note that $\overline{X}' = X'$. So $(\overline{X}, \sigma(\overline{X}, X'))$ is an Ascoli space by Lemma 2.7 of [8]. Therefore the topology $\overline{\tau}$ of \overline{X} coincides with $\sigma(\overline{X}, X')$ by Theorem 5.1. Thus $\tau = \overline{\tau}|_X = \sigma(X, X')$. \square

Since the space φ is barrelled and its topology is strictly stronger than the weak topology, the space φ_w is not Ascoli. Now we prove Corollary 1.8.

Proof of Corollary 1.8. The implications (i) \Rightarrow (ii) and (iv) \Rightarrow (i) are clear. Note that the closure \overline{X} of X is a Fréchet space. So (ii) \Rightarrow (iii) follows from Corollary 5.2. (iii) \Rightarrow (iv) follows from the fact that the Fréchet space \overline{X} coincides with \mathbb{R}^Γ , where Γ is a Hamel base of X' . \square

To prove Theorem 1.9 we need the following lemma.

Lemma 5.3. *Let $\{y_n\}_{n \in \omega}$ be an independent sequence in a locally convex space E . Then for every finite subfamily $\{z_0, \dots, z_m\}$ of E' there are $a_0, \dots, a_{m+1} \in \mathbb{R}$ such that*

$$0 \neq a_0 y_0 + \dots + a_{m+1} y_{m+1} \in \bigcap_{i=0}^m \ker(z_i).$$

Proof. Consider the map $T : \mathbb{R}^{m+2} \rightarrow \mathbb{R}^{m+1}$ defined by

$$T(a_0, \dots, a_{m+1}) := A \cdot (a_0, \dots, a_{m+1}), \text{ where } A := (z_i(y_k))_{i,k}.$$

Since $\ker(T) \neq 0$ there are $a_0, \dots, a_{m+1} \in \mathbb{R}$ such that

$$(a_0, \dots, a_{m+1}) \in \ker(T) \setminus \{0\}.$$

Then the vector $v := a_0 y_0 + \dots + a_{m+1} y_{m+1}$ is as desired. \square

Proof of Theorem 1.9. Suppose for a contradiction that E is infinite dimensional. Therefore there is an independent sequence $\{y_n\}_{n \in \omega}$ of unit vectors in E' . By [15, §20.5(5)], for every $n \in \omega$ there is a closed subspace H_n of the space $E'_{w*} := (E', \sigma(E', E))$ such that

$$E'_{w*} = L_n \oplus H_n, \text{ where } L_n := \text{span}\{y_0, \dots, y_n\} \subseteq E'.$$

For every $n \in \omega$ let $\pi_n : E'_{w*} \rightarrow L_n$ be the continuous projection. As the closed unit ball B of E'_{w*} is compact, there is $b_n > 0$ such that

$$\pi_n(nB) \subseteq [-b_n, b_n]^{n+1}.$$

Take a continuous function $g_n : \mathbb{R}^{n+1} \rightarrow [0, 1]$ such that

$$g_n(x) = 0 \text{ if } x \in [-b_n, b_n]^{n+1}, \text{ and } g_n(x) = 1 \text{ if } x \notin [-b_n - 1, b_n + 1]^{n+1}.$$

Finally we set $f_n := g_n \circ \pi_n, n \in \omega$. To get a contradiction we show that: (1) $f_n \rightarrow 0$ in $C_k(E'_{w*})$, and (2) the sequence $\{f_n\}$ is not equicontinuous at $0 \in E'$.

(1) Let K be a compact subset of E'_{w*} . By the Banach–Steinhaus theorem there is $m \in \omega$ such that $K \subseteq mB$. Now if $n \geq m$ we obtain that $f_n|_K = 0$. So $f_n \rightarrow 0$ in the compact-open topology.

(2) Let U be a standard open neighborhood of zero in E'_{w*} . So there are $\delta > 0$ and $z_0, \dots, z_n \in E$ such that

$$U = \{y \in E' : |y(z_i)| < \delta \forall i = 0, \dots, m\}.$$

Since $\{y_n\}_{n \in \omega}$ is independent we apply Lemma 5.3 to find $a_0, \dots, a_{m+1} \in \mathbb{R}$ such that

$$(5.1) \quad 0 \neq v := a_0 y_0 + \dots + a_{m+1} y_{m+1} \in \bigcap_{i=0}^m \ker(z_i).$$

Choose $\lambda > 0$ such that $\lambda v \notin [-b_{m+1} - 1, b_{m+1} + 1]^{m+2}$. Then $f_{m+1}(\lambda v) = 1$. Since $\lambda v \in U$ by (5.1), we obtain $|f_{m+1}(\lambda v) - f_{m+1}(0)| = 1$, and hence $\{f_n\}$ is not equicontinuous.

Now (1) and (2) show that E'_{w*} is not Ascoli, a contradiction. Thus E is finite dimensional. \square

Remark 5.4. We proved in [10] that a normed space E in the weak topology is Ascoli if and only if E is finite-dimensional using the fact that $(E, \sigma(E, E'))$ has countable fan tightness (this fact was firstly noticed in [11]). The original proof of Corollary 1.8 (see, [2, Theorem 6.1.1]) also essentially uses the same fact. Our proof of Corollary 1.8 is more natural for the theory of locally convex spaces and it does not use specific notions (as fan tightness) from general topology. Also the countability of (fan) tightness cannot be used in the proof of Theorem 1.9 because the tightness of the weak* dual space of a Banach space can be not countable.

REFERENCES

1. T. Banach, \mathfrak{P}_0 -spaces, *Topology Appl.* **195** (2015), 151–173.
2. T. Banach, Fans and their applications in General Topology, Functional Analysis and Topological Algebra, available in arXiv:1602.04857.
3. T. Banach, S. Gabrielyan, On the C_k -stable closure of the class of (separable) metrizable spaces, *Monatshefte Math.* **180** (2016), 39–64.
4. R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.
5. J. Flood, Free locally convex spaces, *Dissertationes Math. CCXXI*, PWN, Warszawa, 1984.
6. S. S. Gabrielyan, The k -space property for free locally convex spaces, *Canadian Math. Bull.* **57** (2014), 803–809.
7. S. S. Gabrielyan, A characterization of free locally convex spaces over metrizable spaces which have countable tightness, *Scientiae Mathematicae Japonicae* **78** (2015), 201–205.
8. S. Gabrielyan, J. Grebík, J. Kąkol, L. Zdomskyy, The Ascoli property for function spaces, *Topology Appl.* **214** (2016), 35–50.
9. S. Gabrielyan, J. Grebík, J. Kąkol, L. Zdomskyy, Topological properties of function spaces over ordinals, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas RACSAM*, accepted.
10. S. Gabrielyan, J. Kąkol, G. Plebanek, The Ascoli property for function spaces and the weak topology of Banach and Fréchet spaces, *Studia Math.* **233** (2016), 119–139.
11. S. Gabrielyan, J. Kąkol, L. Zdomskyy, On topological properties of the weak topology of a Banach space, *J. Convex Anal.* **24** (2017), to appear.
12. S. S. Gabrielyan, S. A. Morris, Free topological vector spaces, submitted.
13. M. Graev, Free topological groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **12** (1948), 278–324 (In Russian). *Topology and Topological Algebra. Translation Series 1*, **8** (1962), 305–364.
14. H. Jarchow, *Locally Convex Spaces*, B.G. Teubner, Stuttgart, 1981.
15. G. Köthe, *Topological vector spaces*, Vol. I, Springer-Verlag, Berlin, 1969.
16. A. A. Markov, On free topological groups, *Dokl. Akad. Nauk SSSR* **31** (1941), 299–301.
17. E. Michael, \aleph_0 -spaces, *J. Math. Mech.* **15** (1966), 983–1002.
18. L. Narici, E. Beckenstein, *Topological vector spaces*, Second Edition, CRC Press, New York, 2011.
19. N. Noble, The continuity of functions on Cartesian products, *Trans. Amer. Math. Soc.* **149** (1970), 187–198.
20. P. J. Nyikos, Metrizability and Fréchet-Urysohn property in topological groups, *Proc. Amer. Math. Soc.* **83** (1981), 793–801.
21. I. Protasov, Maximal vector topologies, *Topology Appl.* **159** (2012), 2510–2512.

- 22. D. A. Raĭkov, Free locally convex spaces for uniform spaces, Math. Sb. **63** (1964), 582–590.
- 23. T. Shirai, Sur les Topologies des Espaces de L. Schwartz, Proc. Japan Acad. **35** (1959), 31–36.
- 24. N. E. Steenrod, A convenient category of topological spaces, Michigan Math. J. **14** (1967), 133–152.
- 25. M. G. Tkachenko, On completeness of free abelian topological groups, Soviet Math. Dokl. **27** (1983), 341–345.
- 26. V. V. Uspenskii, On the topology of free locally convex spaces, Soviet Math. Dokl. **27** (1983), 781–785.
- 27. V. V. Uspenskii, Free topological groups of metrizable spaces, Math. USSR-Izv. **37** (1991), 657–680.

DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER-SHEVA, P.O. 653, ISRAEL
E-mail address: `saak@math.bgu.ac.il`